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On the unitarity of quantum gauge theories on non-commutative spaces

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Abstract: We study the perturbative unitarity of non-commutative quantum Yang-Mills theories, extending previous investigations on scalar field theories to the gauge case where non-locality mingles with the presence of unphysical states. We concentrate our efforts on two different aspects of the problem. We start by discussing the analytical structure of the vacuum polarization tensor, showing how Cutkoski's rules and positivity of the spectral function are realized when non-commutativity does not affect the temporal coordinate. When instead non-commutativity involves time, we find the presence of extra troublesome singularities on the p_0^2-plane that seem to invalidate the perturbative unitarity of the theory. The existence of new tachyonic poles, with respect to the scalar case, is also uncovered. Then we turn our attention to a different unitarity check in the ordinary theories, namely time exponentiation of a Wilson loop. We perform a $\mathrm{O}(g^4)$ generalization to the (spatial) non-commutative case of the familiar results in the usual Yang-Mills theory. We show that exponentiation persists at $\mathrm{O}(g^4)$ in spite of the presence of Moyal phases reflecting non-commutativity and of the singular infrared behaviour induced by UV/IR mixing.

Keywords: Renormalization Regularization and Renormalons, Non-Commutative Geometry.

Contents

1. Introduction

Recently field theories defined on non-commutative spaces have received much attention, mostly triggered by their tight relation with some limiting cases of string theories [1, 2, 3]. These field theories are non local and non-locality has dramatic consequences on their basic dynamical features [4, 5]: although their ultimate "physical" motivation is provided, in our opinion, by their stringy derivation, the possibility of exploring some specific non-local field theories in a concrete, systematic way in search of unexpected properties is fascinating on its own.

Non-commutative field theories are explicitly constructed from the conventional (commutative) ones by replacing the usual multiplication of fields in the lagrangian with the $\star$-product of fields. The $\star$-product is obtained by introducing a real antisymmetric matrix $\theta^{\mu\nu}$ which parametrizes non-commutativity of Minkowski space-time:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}\,, \qquad \mu,\nu = 0,\ldots,D-1\,. \tag{1.1}$$

The $\star$-product of two fields $\phi_1(x)$ and $\phi_2(x)$ is defined as

$$\phi_1(x)\star\phi_2(x) = \exp\left[\frac{i}{2}\,\theta^{\mu\nu}\frac{\partial}{\partial x_1^\mu}\frac{\partial}{\partial x_2^\nu}\right]\phi_1(x_1)\phi_2(x_2)|_{x_1=x_2=x} \tag{1.2}$$

and leads to terms in the action with an infinite number of derivatives of fields which makes the theory non local. Then one may wonder under which conditions the theory would still fulfill the unitarity requirements.

Unitarity is of course a central issue for the correct physical interpretation of a quantum field theory: as is well known, it is granted once the time evolution of a system in a Hilbert space is driven by a self-adjoint operator, its hamiltonian. In the ordinary case the problem becomes subtle when gauge theories are concerned: the norm in the full Fock space is not positive definite, ghosts propagate as virtual states in physical processes and unitarity is formally recovered at the level of the S-matrix looking at transition amplitudes between physical states (selected by the BRST condition). In quantum non-commutative gauge theories, where non-locality mingles with the presence of unphysical states and with the poor definition of S-matrix elements, the investigation of unitarity becomes particularly challenging.

Unitarity of scalar field theories, in the presence of non-commutativity, has been discussed, in a perturbative framework, in ref. [6]: the authors explicitly show that Cutkoski's rules are correct when $\theta^{\mu\nu}$ is of the "spatial" type, i.e. $\theta^{0i} = 0$. This exactly corresponds to the case in which an elegant embedding into string theory is possible: low-energy excitations of a D-brane in a magnetic background are in fact described by field theories with space non-commutativity [3]. In this limit the relevant description of the dynamics is in terms of massless open string states, while massive open string states and closed strings decouple: therefore the full unitary string theory seems consistently truncated to field-theoretical degrees of freedom, suggesting the possibility that also related quantum field theories are unitary. The picture is consistent even at string-loop level as shown in [7].

On the other hand theories with $\theta^{0i} \neq 0$ have an infinite number of time derivatives and are non local in time: in this situation it is not clear whether the usual framework of quantum mechanics makes sense, in particular unitarity may be in jeopardy when the non-commutative parameter $\theta^{\mu\nu}$ affects the time evolution, the concept of hamiltonian losing, in some sense, its meaning (see however [8]).

This fact is not surprising when observed from the string theory point of view: $\theta^{0i} \neq 0$ is obtained in the presence of an electric background and recent works [9] have pointed out that in the relevant low-energy limit massive open string states do not decouple while closed strings do. The truncation of such a string theory to its massless sector is not consistent in this case (see also [10]).

The breakdown of unitarity in time-like scalar non-commutative theories has been recently discussed in refs. [11, 12] in the non-relativistic case.[1]

In this paper we study unitarity properties of non-commutative quantum Yang-Mills theories.

Our first effort (section 2) is to generalize and deepen the work of [6] to the gauge theory case: we concentrate our attention on the vacuum polarization tensor $\Pi^{\mu\nu}$. We derive the complete one-loop result in general $D = 2\omega$ dimensions, using Feynman gauge. Going to $D = 4$ we recover the well-known fact that only planar diagrams are UV-divergent while the non-planar part depends separately on two different kinematical variables, $p^2 = p_\mu p^\mu$ and $\tilde{p}^2 = \theta_{\mu\nu} p^\nu \theta^{\mu\lambda} p_\lambda$. From the point of

[1] The limiting case of a light-like $\tilde{p}_\mu$ has been discussed in ref. [13]. The authors show that unitarity constraints are fulfilled also in this situation, provided the theory is formulated in terms of a light-front quantization.

view of the dispersion relations the situation is clear in the non-commutative spatial case: the vector $\tilde{p}_\mu$ is spacelike and there is no point in analytically continuing the variable $\tilde{p}^2$. We have to deal with normal dispersion relations in p^2 with the obvious presence of an extra kinematical variable.

The situation is much trickier when non-commutativity involves time. Then $\tilde{p}_\mu$ can also be timelike and it is not clear *a priori* which variable should be analytically continued. The natural choice would be p_0^2 in this case, also in view of the fact that the Lorentz invariance is broken in such theories. However, even with this choice, Cutkoski's rules are still invalid, as already pointed out in ref. [11] for the scalar case; the presence of extra troublesome singularities in the p_0^2-plane, while being an obvious sign of instability of the theory, can hardly be explained in a perturbative context. Moreover, both in the spatial and in the space-time non-commutative cases, new poles appear in the one-loop resummed vector propagator for negative values of p^2 (tachyons). All these features are described in section 3.

In ordinary theories, another typical probe to check unitarity is provided by time exponentiation of a Wilson loop. To be more specific, for a rectangular loop centered in the plane (t, x), x being any spatial direction, with sides $2T$ and $2L$, one can show that the Wilson loop amplitude exponentially decreases in the large-T limit, and the exponent is related to the potential energy of a (very heavy) $q\bar{q}$ pair separated by a distance $2L$. A perturbative computation of the Wilson loop has been widely used in commutative theories to check unitarity, assuming gauge invariance, or viceversa [14].

To extend this test to non-commutative theories is highly problematic, even in the spatial case. As a matter of fact the definition of the loop via a non-commutative path-ordering [15, 16, 17], has so far received a physical interpretation in the presence of matter fields as a wave function of composite operators only in a lattice formulation [18]. Time exponentiation itself has not been proven to our knowledge, even in the spatial case.

Section 4 is devoted to a perturbative $\mathcal{O}(g^4)$ generalization to the spatial non-commutative case of the familiar results in the usual theory. We show that exponentiation persists at $\mathcal{O}(g^4)$ in spite of the phases which reflect non-commutativity.

Finally, in section 5 we draw our conclusions and discuss future developments, whereas technical details are deferred to appendix A.

2. U(N) non-commutative Yang-Mills

In this section we analyze the U(N) Yang-Mills theory on a non-commutative space. The classical action reads

$$S = -\frac{1}{2g^2} \int d^4x \, F_{\mu\nu} \star F^{\mu\nu} \,, \tag{2.1}$$

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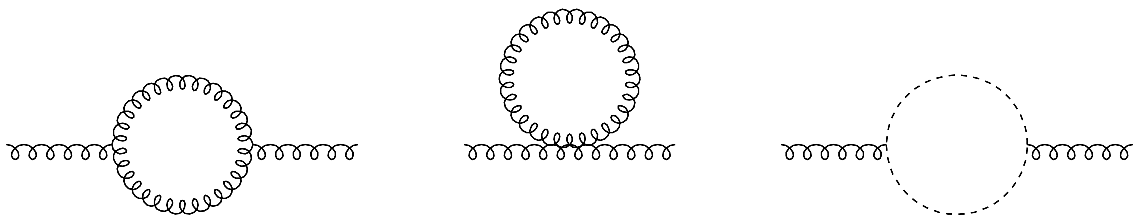

Figure 1: One-loop corrections to the two-point function.

where the field strength $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\left(A_\mu \star A_\nu - A_\nu \star A_\mu\right) \tag{2.2}$$

and A_μ is a $N \times N$ matrix. The $\star$-product was defined in eq. (1.2). The action eq. (2.1) is invariant under $\mathrm{U}(N)$ non-commutative gauge transformations

$$\delta_\lambda A_\mu = \partial_\mu \lambda - i\left(A_\mu \star \lambda - \lambda \star A_\mu\right). \tag{2.3}$$

The Feynman rules for a non-abelian non-commutative gauge theory were worked out in [19] and a full list is reported in appendix A, together with our conventions; for further investigations on quantum aspects of non-commutative gauge theories see [20]. We quantize the theory in the Feynman-'t Hooft gauge and, in order to check unitarity and gauge invariance at the quantum level, we consider the one-loop correction to the gluon self energy. One-loop diagrams contributing to the two-point function are shown in figure 1. In the ordinary gauge theories, tadpole diagrams vanish in dimensional regularization ($D = 2\omega = 4 - 2\varepsilon$). However, this is not true in the non-commutative case and the tadpole must be included in the computation. By using the Feynman rules given in appendix A the sum of the diagrams of figure 1 turns out to be

$$\begin{aligned}\Pi^{AB}_{\mu\nu}(p) = N(g\,\mu^{2-\omega})^2 \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \Bigg[&\frac{4(p^2 g_{\mu\nu} - p_\mu p_\nu)}{q^2(p-q)^2} + \\ &+ 2(\omega-1)\left(\frac{(p-2q)_\mu (p-2q)_\nu}{q^2(p-q)^2} - \frac{2g_{\mu\nu}}{q^2}\right)\Bigg] \times \\ &\times \left(\delta^{AB} - \delta^{A0}\delta^{B0}\cos(\tilde{p}q)\right),\end{aligned} \tag{2.4}$$

where $\tilde{p}^\mu = \theta^{\mu\nu} p_\nu$ and $\theta^{12} = -\theta^{21} \equiv \theta$, all the other components vanishing. One immediately recognizes that in eq. (2.4) the planar and the non-planar (i.e. θ-dependent) contributions can be singled out. The term proportional to δ^{AB} corresponds to the planar diagrams [4], and coincides with ordinary Yang-Mills theory with the usual $\mathrm{U}(N)$ group factor $N\delta^{AB}$. In four dimensions this integral is divergent and produces, once regulated, the well-known $1/\varepsilon$ pole. On the other hand all the novelty of non-commutativity is concentrated in the term with $\cos(\tilde{p}q)$, corresponding to the ultraviolet finite non-planar contribution. Since this term only affects the $\mathrm{U}(1)$

propagator, in the following we will restrict ourselves, with no loss of generality, to the U(1) case, where $\Pi_{\mu\nu}$ becomes

$$\Pi_{\mu\nu}(p) = (g\,\mu^{2-\omega})^2 \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \left[\frac{4\,(p^2 g_{\mu\nu} - p_\mu p_\nu)}{q^2(p-q)^2} + \right.$$
$$\left. + 2(\omega-1)\left(\frac{(p-2q)_\mu (p-2q)_\nu}{q^2(p-q)^2} - \frac{2g_{\mu\nu}}{q^2}\right)\right] \times$$
$$\times\,(1-\cos(\tilde{p}q))\,. \tag{2.5}$$

One can easily realize that this tensor is orthogonal to p_μ and thereby can be written as

$$\Pi_{\mu\nu} = \left(g_{\mu\nu}p^2 - p_\mu p_\nu\right)\Pi_1 + \tilde{p}_\mu \tilde{p}_\nu \Pi_2\,. \tag{2.6}$$

In turn Π_1 contains the usual planar part and a non-planar one

$$\Pi_1 = \Pi_1^p + \Pi_1^{np}\,.$$

Only Π_1^p is ultraviolet divergent and therefore needs to be (dimensionally) regularized; on the other hand Π_1^{np} and Π_2 exhibit singularities when $\tilde{p}^2 = 0$.

Standard Feynman diagram techniques lead to the results

$$\Pi_1^p = \frac{i\,g^2}{16\pi^2}\left[\frac{-p^2}{4\pi\mu^2}\right]^{\omega-2} \frac{6\omega-2}{2\omega-1}\,\frac{\Gamma(2-\omega)\Gamma^2(\omega-1)}{\Gamma(2\omega-2)}\,, \tag{2.7}$$

$$\Pi_1^{np} = \frac{-i\,g^2}{4\pi^2}\left[\frac{p^2}{16\pi^2\mu^4\tilde{p}^2}\right]^{\frac{\omega}{2}-1} \int_0^1 dx[x(1-x)]^{\frac{\omega}{2}-1}[2-(\omega-1)(1-2x)^2] \times$$
$$\times\,K_{2-\omega}\left(\sqrt{x(1-x)p^2\tilde{p}^2}\right) \tag{2.8}$$

and

$$\Pi_2 = \frac{i\,g^2}{4\pi^2}[4\pi\mu^2]^{2-\omega}\left[\frac{4p^2}{\tilde{p}^2}\right]^{\omega/2} \int_0^1 dx[x(1-x)]^{\omega/2} K_\omega\left(\sqrt{x(1-x)p^2\tilde{p}^2}\right). \tag{2.9}$$

It is easy to recognize from the above formulas that, at fixed $\theta^2 p_\perp^2 = \theta^2(p_1^2+p_2^2) = -\tilde{p}^2 > 0$, the components of the polarization tensor are analytic functions of the variable p^2, with a branch point at $p^2=0$ and a cut that can be conveniently drawn along the positive p^2-axis. It is also straightforward to compute their discontinuities

$$\Delta\Pi_1^p = -\frac{5g^2}{12\pi}\theta(p^2)\,\theta(p^0)\,,$$
$$\Delta\Pi_1^{np} = \frac{g^2}{4\pi}\theta(p^2)\,\theta(p^0)\int_0^1 dx\,[1+4x(1-x)]\,J_0\left(\theta\sqrt{x(1-x)\,p^2p_\perp^2}\right) \tag{2.10}$$

and

$$\Delta\Pi_2 = -\frac{g^2p^2}{\pi\,\theta^2p_\perp^2}\theta\left(p^2\right)\,\theta\left(p^0\right)\int_0^1 dx\,x(1-x)\,J_2\left(\theta\sqrt{x(1-x)\,p^2p_\perp^2}\right), \tag{2.11}$$

where we have now set $\omega=2$.

These discontinuities can also be computed applying Cutkoski's cutting rules to eq. (2.5)

$$\Delta\Pi_{\mu\nu}(p) = -g^2 \int \frac{d^4q}{(2\pi)^2} \left[4\left(p^2 g_{\mu\nu} - p_\mu p_\nu\right) + 2(p-2q)_\mu (p-2q)_\nu\right] \delta\left(q^2\right)\, \theta\left(q^0\right) \times \\ \times\, \delta\left((p-q)^2\right)\, \theta\left(p^0 - q^0\right)(1-\cos(\tilde{p}q))]. \tag{2.12}$$

We have checked that indeed the cutting rules hold also in this non-commutative context, provided the non-commutative parameter $\theta_{\mu\nu}$ has only spatial components. This claim has already been presented for the scalar theory in refs. [6, 11]. In the stringy context, this picture has its counterpart in the decoupling of massive open and closed string states in the presence of a magnetic background.

The analytic properties outlined above are necessary requirements for fulfilling unitarity; next, positivity conditions, crucial for the correct probabilistic interpretation of the theory, are to be met. When saturating the tensor (2.5) with a (spacelike) polarization vector ε^μ and computing its discontinuity afterwards, a non-negative result is expected since, in the one-loop case we are considering, positivity cannot be spoilt by the extra phases affecting the vertices. A convenient choice is $\varepsilon_\mu = \tilde{p}_\mu/\sqrt{-\tilde{p}^2}$, which gives

$$\begin{aligned} \Delta\left(\varepsilon\Pi\varepsilon\right) &= \frac{g^2p^2}{4\pi}\left[\frac{5}{3} - \int_0^1 dx\,[1+4x(1-x)]\, J_0\left(\theta\sqrt{x(1-x)\,p^2p_\perp^2}\right) + \right. \\ &\qquad \left. + 4\int_0^1 dx\, x(1-x)\, J_2\left(\theta\sqrt{x(1-x)\,p^2p_\perp^2}\right)\right] \\ &= \frac{g^2p^2}{4\pi}\left[\frac{5}{3} - 3\frac{\sin\xi}{\xi} + 4\frac{\sin\xi - \xi\cos\xi}{\xi^3}\right] \geq 0\,, \qquad \xi \equiv \frac{\theta\sqrt{p^2p_\perp^2}}{2}\,. \end{aligned} \tag{2.13}$$

The function in square brackets vanishes only at $\xi = 0$ and for $\xi > 0$ is indeed positive.

Such a positivity is strongly reminiscent of the usual behaviour of abelian theories defined on commutative spaces. To clarify this issue, let us briefly recall the general expression for the polarization tensor $\Pi_{\mu\nu}$ in those theories

$$\Pi_{\mu\nu}(p) = i\int d^4x\, e^{ip(x-y)}\langle 0|T\left(\mathcal{J}_\mu(x)\,\mathcal{J}_\nu(y)\right)|0\rangle + \text{contact terms}\,, \tag{2.14}$$

the current $\mathcal{J}_\mu$ being the source of the vector field and T the Dyson time-ordering operator. Current conservation implies that $\Pi_{\mu\nu}$ is transverse with respect to p_μ. If we saturate the tensor (2.14) with a (spacelike) polarization vector ε^μ, we get, for its discontinuity, the well-known result

$$\Delta(\varepsilon\Pi\varepsilon) \propto \sum_n |\langle 0|\varepsilon\mathcal{J}(0)|n\rangle|^2 \delta^{(4)}\left(p - P_n\right), \tag{2.15}$$

the vector P_n being the total four-momentum of the on-shell intermediate state $|n\rangle$. In a gauge theory some intermediate states may possess a negative norm; nevertheless their presence is necessary to cancel possible redundant degrees of freedom in such a way that "physical" positivity is eventually recovered.

However in the non-commutative case novel features arise: in fact, in eq. (2.13) the asymptotic value 5/3 of the planar contribution is reached after wiggling: this feature, which is present also in scalar theories [6, 11], reminds of analogous quantum mechanical effects (diffraction and interference) and is a consequence of the non-commutativity of coordinates, which in turn entails a violation of locality and of Lorentz invariance.

The vanishing of the discontinuity at $p^2 = 0$ can be understood as a threshold effect; nonetheless, $\xi = 0$ can also mean $p_\perp = 0$. The behaviour of the theory in this limit looks peculiar and will be discussed later on. Suffice it here to say that in the reference frame and in the approximation we are considering, no production occurs in the plane $p_\perp = 0$. This *non-absorbing* phase of the theory is a novel feature compared to the commuting case. On the other hand $\Pi_{\mu\nu}$ itself is singular in such a limit and cannot be cured.

In general non-local theories lead to amplitudes which are not polynomially bounded at infinity on the first Riemann sheet; this issue stands at the very heart of a renormalization program and deserves thorough investigations.

In view of the non-locality of the theory, one might wonder whether amplitudes can still be reconstructed from their imaginary parts via dispersion relations. In the instances we are considering the answer is affirmative, but to a certain extent. The discontinuity $\Delta\Pi_1^p$ is constant and therefore Π_1^p needs to be subtracted once, as is well-known; $\Delta\Pi_1^{np}$ vanishes at infinity and no subtraction is *a priori* needed. The situation with Π_2 is subtler and somehow pathological. In commutative theories subtraction constants in dispersion relations are related to the presence of singularities in Feynman amplitudes. The arbitrariness of such constants is in turn related to different renormalization prescriptions. Thus, increasing discontinuities usually correspond to divergent amplitudes. This is not always the case in non-commutative theories; as a matter of fact the absolute value of $\Delta\Pi_2$ increases at infinity in four dimensions. As a consequence the dispersion relation needs to be subtracted (once). Nevertheless we know from Feynman diagram calculations that Π_2 is finite when $\omega = 2$ and therefore the subtraction constant turns out to be completely determined.

One could consider the dispersion relation starting from the discontinuity in 2ω dimensions; then, for suitable values of ω, no subtraction is necessary and, at variance with the usual case, no pole occurs in the continuation to $\omega = 2$. However this procedure is somehow extraneous to the spirit of a dispersive approach, being possible only in a perturbative context where amplitudes can be directly computed anyway.

In recovering eqs. (2.8) and (2.9) from eqs. (2.10) and (2.11), respectively, one may use the Stieltjes transform

$$\int_0^\infty \frac{dx}{x+y} J_0\left(a\sqrt{x}\right) = 2\, K_0\left(a\sqrt{y}\right) , \qquad a > 0 \, , \tag{2.16}$$

together with the equalities

$$2\int_0^1 dx\, x(1-x)\, J_2\left(\theta\sqrt{x(1-x)\,p^2p_\perp^2}\right) = \int_0^1 dx\,[1-6x(1-x)]\times \times J_0\left(\theta\sqrt{x(1-x)\,p^2p_\perp^2}\right) \tag{2.17}$$

and

$$2\int_0^1 dx\, x(1-x)\, K_2\left(\theta\sqrt{-x(1-x)\,p^2p_\perp^2}\right) = -\frac{4}{\theta^2p^2p_\perp^2} - \int_0^1 dx[1-6x(1-x)]\times \times K_0\left(\theta\sqrt{-x(1-x)\,p^2p_\perp^2}\right). \tag{2.18}$$

It is technically difficult to control the asymptotic behaviour of Π_1^{np} and Π_2 in the variable p^2 on the entire first Riemann sheet. One can show that they both vanish as $(-p^2)^{-1}$ when $p^2 \to -\infty$ along the real half-line, in spite of the fact that $\Delta\Pi_2$ diverges when $p^2 \to +\infty$.

The coincidence between eqs. (2.8), (2.9) and the results one obtains via dispersion relations starting from eqs. (2.10), (2.11) is a proof *a posteriori* that indeed their asymptotic behaviour is compatible with discarding the contribution to the dispersive integrals at infinity.

3. IR properties and tachyons

We now comment on the infrared singularities, namely the ones at small $\tilde{p}^2$, affecting Π_1^{np} and Π_2. As is well known, they are the counterparts of the would-be ultraviolet singularities in the absence of the non-commutative phase. Although quite clear from a mathematical viewpoint, their presence is particularly troublesome in higher order calculations where the momentum p has to be integrated over. For a proposal of using Wilsonian methods to tame infrared singularities and to prove UV renormalizability, in the scalar case, see [21].

If we recall the Dyson equation for the renormalized vector propagator

$$\left(D_{\mu\nu}^{(\mathrm{ren})}\right)^{-1} = \left(D_{\mu\nu}^{(0)}\right)^{-1} - \Pi_{\mu\nu}^{(\mathrm{ren})}\,, \tag{3.1}$$

where $D_{\mu\nu}^{(0)}$ is the free propagator, it is clear that infrared divergences of $\Pi_{\mu\nu}^{(\mathrm{ren})}$ cannot affect $D_{\mu\nu}^{(\mathrm{ren})}$. Although certainly appearing order by order in the perturbative expansion, they might be artifact of this expansion. On the other hand at two loops and beyond, infrared divergencies other than iterated one-loop ones, may be generated and their explicit form has to be taken into account in order to draw conclusions on the finiteness of $D_{\mu\nu}^{(\mathrm{ren})}$ at small $\tilde{p}^2$.

Moreover the very possibility of a renormalization in a non-commutative gauge theory has not been proved beyond one loop, to our knowledge.

The vector propagator $D^{(\mathrm{ren})}_{\mu\nu}$ acquires the analytic structure induced by $\Pi^{(\mathrm{ren})}_{\mu\nu}$; it is an analytic function in the cut p^2-plane with possible simple poles at negative values of p^2. However such poles (tachyons) would conflict with causality and signal instability of the theory. If present, they may give rise to spontaneous symmetry breaking after condensation. Moreover in this context they look dependent on the gauge choice, on the running mass μ and on the renormalization scheme. All these dependences should eventually disappear in any realistic solution.

In particular poles of the vector propagator on the negative p^2-axis at a certain loop order might indicate that the perturbative approach fails at some momentum scale and that non-perturbative effects may change the infrared behaviour of the theory.

In the Feynman gauge we are considering, taking the one-loop renormalized expression for Π_1 in the $\overline{\mathrm{MS}}$ scheme

$$\Pi_1^{(\mathrm{ren})} = -\frac{ig^2}{16\pi^2}\frac{10}{3}\log\left(\frac{-p^2}{4\pi\mu^2}\right) - \frac{ig^2}{4\pi^2}\int_0^1 dx\,[1+4x(1-x)]\,K_0(\sqrt{x(1-x)p^2\tilde{p}^2}) \quad (3.2)$$

into account, eq. (3.1) can be inverted, leading to

$$D^{(\mathrm{ren})}_{\mu\nu} = -\frac{i}{p^2\left(1+i\Pi_1^{(\mathrm{ren})}\right)}\left[g_{\mu\nu} + i\Pi_1^{(\mathrm{ren})}\frac{p_\mu p_\nu}{p^2} - \frac{i\,\Pi_2}{p^2\left(1+i\Pi_1^{(\mathrm{ren})}\right)+i\tilde{p}^2\Pi_2}\tilde{p}_\mu\tilde{p}_\nu\right]. \quad (3.3)$$

One easily realizes from eq. (3.2) that $\Pi_1^{(\mathrm{ren})}$ is finite at $p^2 = 0$; therefore, after the usual subtraction in the planar contribution, the residue on the pole at $p^2 = 0$ is changed only by a finite amount and the logarithmic branch point exhibits a "mild" behaviour. We remark that in usual Yang-Mills theory $\Pi_1^{(\mathrm{ren})}$ is not finite as $p^2 \to 0$: the present behaviour is a pure non-commutative effect, that can be easily understood realizing that, in that limit, planar and non-planar contributions, in the U(1) case, conspire to cancel thanks to IR/UV duality (see [22] for a discussion on this point). At $\tilde{p}^2 = 0$, the "soft" singularity of $\Pi_1^{(\mathrm{ren})}$ and the "hard" one of Π_2 are completely sterilized by the one-loop resummation, as expected.

Other possible singularities come from the vanishing of the other two denominators. The vanishing of $(1 + i\,\Pi_1^{(\mathrm{ren})})$ would correspond to Landau poles and looks dependent on the gauge choice and on the subtraction procedure. It could not occur for small values of the coupling constant, nor of $\tilde{p}^2$. Much more interesting is the possibility of a vanishing of the second denominator, which can occur also for small g^2, though being a typically non-perturbative effect. If we neglect the contribution from $\Pi_1^{(\mathrm{ren})}$, we have the condition

$$\frac{g^2}{\pi^2}\int_0^1 dx\,x(1-x)\,K_2\left(\sqrt{x(1-x)p^2\tilde{p}^2}\right) = 1\,. \quad (3.4)$$

In the spatial case ($\theta_{12} = \theta$) we have hitherto considered, $\tilde{p}^2 = -\theta^2 p_\perp^2$ and, as already noticed in refs. [4, 22], a pole at the value $p^2 = -\gamma^2/p_\perp^2$, $\gamma^2 \equiv 2g^2/(\pi^2\theta^2)$, appears, for $g^2 \ll 1$ in the approximation of retaining only the leading term in K_2. Close to this pole we find the behaviour

$$D_{\mu\nu} \approx \frac{i\,\varepsilon_\mu\,\varepsilon_\nu}{p^2 + \frac{\gamma^2}{p_\perp^2}}\,, \tag{3.5}$$

exhibiting a residue independent of γ^2 and of $p_\perp^2$ and with the correct sign. This solution does not depend on the choice of the gauge parameter, as pointed out in ref. [22], and, at least in the one-loop approximation, seems to support the existence of such an instability. Similar phenomena have also been observed in scalar theories [23]. It is present only in one component of the tensor, the violation of Lorentz invariance allowing for different dispersion relations in different projections and it is related to the would-be quadratic mass divergence in the usual theory; as such it is expected to be gauge invariant and disappearing in the supersymmetric case [5, 24]. It might be interesting to check its properties in the one-loop expression of the four-vector amplitude.

We now discuss the unitarity implications of when the non-commutative parameter involves the time direction, namely $\theta_{03} = -\theta_{30} = \theta$. Some results in scalar theories have been reported in ref. [11] together with their interpretation in connection with string theory.

By repeating the Feynman diagram calculation one recovers the decomposition (2.6) and eqs. (2.7), (2.8) and (2.9). However one has to keep in mind that now the vector $\tilde{p}_\mu$ can also be timelike. There is therefore a delicate problem in analytically continuing eqs. (2.8) and (2.9). We can still consider the variable p^2 taking the relation $\tilde{p}^2 = -\theta^2(p^2 + p_\perp^2)$ into account, with $p_\perp^2$ being kept fixed at positive values.

By looking at eqs. (2.8) and (2.9) one can easily realize the presence of two branch points at the values $p^2 = 0$ and $p^2 = -p_\perp^2$. The amplitudes are real in the gap $-p_\perp^2 < p^2 < 0$. The right-hand cut is referred to as the usual "physical" cut, whereas the cut for negative values is the non-commutative one occurring when the parameter θ has a time component (electric case). A natural interpretation in terms of Cutkoski's rules is available for the "physical" cut, whereas extra tachyonic excitations should be invoked to explain the presence of the other threshold [11].

Finally we comment on the presence of possible bound states in this case. In the same approximation of eq. (3.4) ($g^2 \ll 1$), we have to solve the quadratic equation

$$p^4 + p^2 p_\perp^2 + \gamma^2 = 0\,, \qquad \gamma^2 = \frac{2g^2}{\pi^2\,\theta^2}\,. \tag{3.6}$$

The solutions are

$$p_\pm^2 = \frac{1}{2}\left[-p_\perp^2 \pm \sqrt{p_\perp^4 - 4\gamma^2}\right]. \tag{3.7}$$

When $p_\perp^2 > 2\gamma$, the roots are real, in the gap between $-p_\perp^2$ and 0. They represent a couple of tachyons:

$$D_{\mu\nu}^{\pm} \approx \pm \frac{i\,\varepsilon_\mu\,\varepsilon_\nu}{p^2 - p_\pm^2}\,\frac{p_\perp^2 \pm \sqrt{p_\perp^4 - 4\gamma^2}}{2\sqrt{p_\perp^4 - 4\gamma^2}}\,. \tag{3.8}$$

In the limit $\gamma^2 \to 0$ ($p_+^2 \to 0\,, p_-^2 \to -p_\perp^2$)

$$D_{\mu\nu}^{+} \approx \frac{i\,\varepsilon_\mu\,\varepsilon_\nu}{p^2}\,, \tag{3.9}$$

and the (+)-pole is turned into a finite correction to the free pole, the same as in the one in the spatial case (see eq. (3.5)), whereas

$$D_{\mu\nu}^{-} \approx -\frac{i\,\varepsilon_\mu\,\varepsilon_\nu}{p^2 + p_\perp^2}\,\frac{\gamma^2}{p_\perp^2} \tag{3.10}$$

and the (−)-pole decouples.

When instead $p_\perp^2 < 2\gamma$ the roots migrate to complex conjugate values and their interpretation looks obscure.

4. Time exponentiation of a Wilson loop as a test of unitarity

A fairly general non-perturbative test of unitarity in the usual commutative case is provided by the time exponentiation of a Wilson loop [25, 26]. To be more specific, one considers a rectangular Wilson loop, centered in the origin of the (t, x)-plane with sides of length $2T$ and $2L$, respectively. One can show that, in the large-T limit, its expression coincides (apart from a trivial threshold factor) with the vacuum-to-vacuum overlap amplitude of two $q\bar{q}$ strings at times $-T$ and T, respectively. The (very heavy) quarks are kept at a fixed finite distance $2L$.

By expanding on the complete set of energy eigenfunctions, after time translations, the loop acquires the expression (for euclidean time)

$$\mathcal{W}(T, L) = \exp(-2\mathcal{E}_0 T) \int_{\mathcal{E}_0}^{\infty} d\mathcal{E}\; \rho(\mathcal{E}, L)\; \exp[-2T(\mathcal{E} - \mathcal{E}_0)]\,, \tag{4.1}$$

$\mathcal{E}_0(L)$ being the ground state energy of the $q\bar{q}$ system. The spectral density ρ is a positive measure, as a consequence of unitarity.

The above equation implies an exponential decrease of the Wilson loop with time. If $\mathcal{E}_0(L)$ increases linearly at large L ($\mathcal{E}_0(L) \simeq 2\sigma L$), an area-law behaviour is obtained. The $q\bar{q}$-potential confines with a string tension σ.

Although the above arguments are non perturbative, a perturbative analysis of the Wilson loop is interesting on its own. The perturbative Wilson loop manifests a feature, which is known in the literature as the non-abelian cancellation of the $\mathcal{O}(g^4)$ T^2-terms. This property is compatible, at $\mathcal{O}(g^4)$, with its perturbative exponentiation in the large-T limit.

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In this section we shall scrutinize the perturbative $\mathcal{O}(g^4)$ large-T behaviour of the Wilson loop in the spatial non-commutative case and compare the results with those of the corresponding commutative theory, although no exponentiation property has been proven to our knowledge nor any connection with a possible $q\bar{q}$-potential established, at least in a continuum formulation.

In the non-commutative case the Wilson loop can be defined by means of the Moyal product as [15, 16]

$$\mathcal{W}[C] = \int \mathcal{D}A\, e^{iS[A]} \int d^4x \,\mathrm{Tr}\, P_\star \exp\left(i \int_C A_\mu(x+\xi(s))\, d\xi^\mu(s) \right), \tag{4.2}$$

where C is a closed contour in non-commutative space-time parametrized by $\xi(s)$, with $0 \le s \le 1$, and $P_\star$ denotes non-commutative path ordering along $x(s)$ from right to left with respect to increasing s of $\star$-products of functions. Gauge invariance requires integration over coordinates, which is trivially realized when considering vacuum averages [17].

We consider the closed path C parametrized by the following four segments γ_i

$$\begin{aligned}
\gamma_1 : \gamma_1^\mu(s) &= (-sT, L)\,, \\
\gamma_2 : \gamma_2^\mu(s) &= (-T, -sL)\,, \\
\gamma_3 : \gamma_3^\mu(s) &= (sT, -L)\,, \\
\gamma_4 : \gamma_4^\mu(s) &= (T, sL)\,, \qquad -1 \le s \le 1\,,
\end{aligned} \tag{4.3}$$

describing a (counterclockwise-oriented) rectangle centered at the origin of the plane $(x_0,\, x_3)$, with length sides ($2T$ and $2L$), respectively. The perturbative expansion of $\mathcal{W}[C] = \mathcal{W}(T, L)$, expressed by eq. (4.2), reads (for a U(1) non-commutative theory)

$$\begin{aligned}
\mathcal{W}(T, L) &= \langle 0|\mathcal{T}(P_\star e^{ig\int_C A_\mu dx^\mu})|0\rangle \\
&= \sum_{n=0}^{\infty} (ig)^n \int_{-1}^{1} ds_1 \ldots \int_{s_{n-1}}^{1} ds_n\, \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n} \times \\
&\qquad\qquad \times\ \langle 0\,|\mathcal{T}\,[A_{\mu_1}(x(s_1)) \star \ldots \star A_{\mu_n}(x(s_n))]|\,0\rangle
\end{aligned} \tag{4.4}$$

and it is easily shown to be an even power series in g, so that we can write

$$\mathcal{W}(T, L) = 1 + g^2\mathcal{W}_2 + g^4\mathcal{W}_4 + \mathcal{O}\left(g^6\right). \tag{4.5}$$

Through an explicit evaluation one is convinced that the function $\mathcal{W}_2$ in eq. (4.5) is reproduced by the single-exchange diagram (figure 2), which is exactly as in the ordinary U(1) theory.

On the other hand the diagrams contributing to $\mathcal{W}_4$ can be grouped into three distinct families:

- those with a double vector exchange in which propagators either do not cross ($\mathcal{W}_{\rm nc}$) or cross ($\mathcal{W}_{\rm c}$);

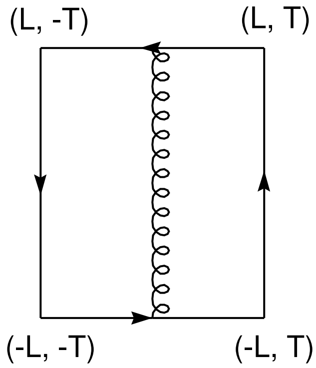


Figure 2: Single exchange.

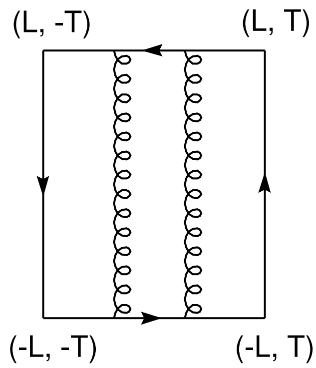


Figure 3: Non-crossed vector exchange.

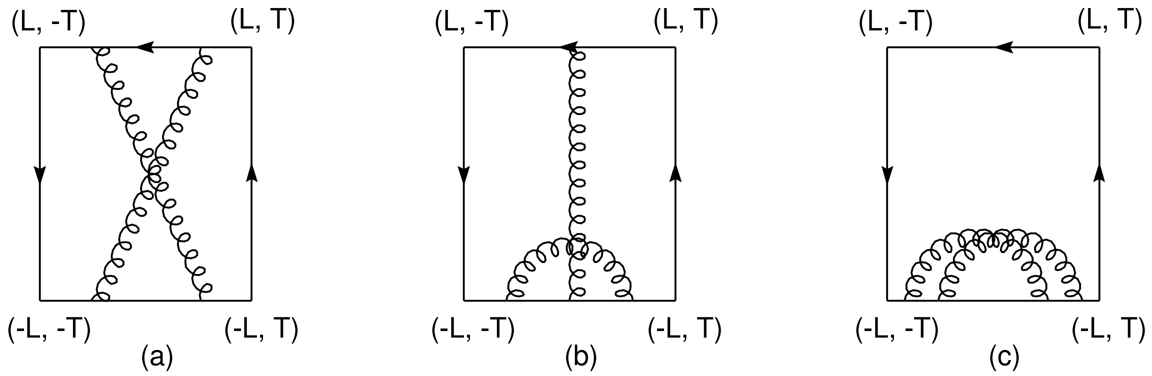


Figure 4: Dominant crossed vector exchanges.

- those involving a vertex ($\mathcal{W}_s$);
- those with a one-loop self-energy insertion in the free propagator ($\mathcal{W}_b$).

In the large-T limit the leading contribution to $\mathcal{W}_{\rm nc}$ is depicted in figure 3 and is given by

$$
\begin{aligned}
\mathcal{W}_{\rm nc}(T,L) &= -\frac{g^4}{(2\pi)^4}\int \frac{d^4p}{p^2}\,\frac{d^4q}{q^2}\int_{-T}^{T}dx\int_{x}^{T}ds\int_{T}^{-T}dw\int_{w}^{-T}dy\; e^{i[p_0(x-y)+q_0(s-w)]}\;\times \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\; e^{2iL(p_3+q_3)} \\
&\approx -\frac{g^4T^2}{\pi^2}\left(\int d^3p\frac{e^{2ip_3L}}{\vec{p}^{\,2}}\right)^2 = -g^4\pi^2\left(\frac{T}{L}\right)^2. \qquad (4.6)
\end{aligned}
$$

We derived such an expression by first rescaling the variables x, y, s, w and then neglecting terms like $(p_0/T)^2$, $(q_0/T)^2$ with respect to $\vec{p}^{\,2}$ and $\vec{q}^{\,2}$ [27].

Our goal is to prove that eq. (4.6) represents the only leading contribution to $\mathcal{W}_4$ also in the non-commutative case. Hence we proceed examining $\mathcal{W}_c$. The potentially $\mathcal{O}(T^2)$ crossed diagrams are those shown in figure 4 together with their symmetric under the exchange $\gamma_1 \to \gamma_3$. As a representative, let us consider the graph depicted in figure 4(a), having the expression

$$
\begin{aligned}
\mathcal{W}_c(T,L) = & -\frac{g^4}{(2\pi)^4}\int \frac{d^4p}{p^2}\,\frac{d^4q}{q^2}\int_{-T}^{T}dx\int_{x}^{T}ds\int_{T}^{-T}dy\,\times \\
& \times\int_{y}^{-T}dw\,e^{i\tilde{p}q}\,e^{i[p_0(x-y)+q_0(s-w)]}e^{2iL(p_3+q_3)} \qquad (4.7)\\
\approx & \frac{g^4T^2}{4\pi^4}\int dp_0\,\frac{dq_0}{q_0^2}\left[e^{iq_0}\frac{\sin(p_0)}{p_0}-\frac{\sin(p_0+q_0)}{p_0+q_0}\right]^2\int d^3p\,d^3q\,e^{2i(p_3+q_3)L}\,\frac{e^{i\tilde{p}q}}{\vec{p}^{\,2}\vec{q}^{\,2}}\,.
\end{aligned}
$$

Integrations over q_0 and p_0 are easily proven to produce a vanishing result. Similarly one can treat integrals arising from graphs in figures 4(b) and (c).

The quantity $\mathcal{W}_s$ comes from "spider" diagrams, namely those containing the triple vector vertex. It can be straightforwardly checked that spider diagrams are at most $\mathcal{O}(T^0)$. By denoting by S_{ijk} the contribution of the diagram in which the vectors are attached to the lines γ_i, γ_j and γ_k, respectively, one has for instance (see figure 5)

$$
\begin{aligned}
S_{233} = & \frac{g^4}{(2\pi)^4}\int d^4p\,d^4q\int_{-T}^{T}dx\int_{x}^{T}dy\,\times \\
& \times\int_{L}^{-L}ds\,V_{003}(p,q,k;\theta)e^{\frac{i}{2}\tilde{p}q}\,e^{ip_0x}\,\times \\
& \quad\times\,e^{iq_0y}\,e^{-ik_0T}e^{-ik_3s}\,e^{iL(p_3+q_3)}\,, \qquad (4.8)
\end{aligned}
$$

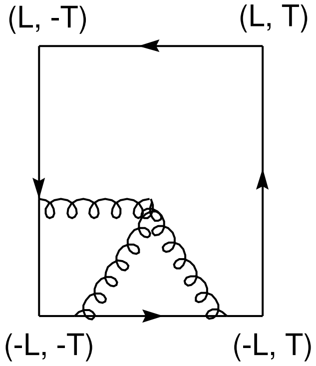


Figure 5: The triple vertex diagram S_{233}.

where

$$
V_{003}(p,q,k;\theta) = 2\,\frac{(p_3-q_3)\sin\left(\tilde{p}q/2\right)}{p^2\,q^2\,k^2}
$$

and $k=-p-q$. After integrating out the geometric variables x and y and adopting the approximation introduced in (4.6), eq. (4.8) reduces to

$$
\begin{aligned}
S_{233} \approx & \frac{i\,g^4}{4\pi^4}\int dp_0\,dq_0\int_{L}^{-L}ds\int\frac{d^3p\,d^3q}{\vec{p}^{\,2}\,\vec{q}^{\,2}\,(\vec{p}+\vec{q})^{\,2}}\,\frac{e^{i(p_0+q_0)}}{q_0}\,\times \\
& \times\left[e^{iq_0}\frac{\sin(p_0)}{p_0}-\frac{\sin(p_0+q_0)}{p_0+q_0}\right]e^{\frac{i}{2}\tilde{p}q}\,e^{i(p_3+q_3)L}\,e^{-ik_3s}\,(p_3-q_3)\,\sin\left(\frac{\tilde{p}q}{2}\right) \qquad (4.9)
\end{aligned}
$$

which is manifestly $\mathcal{O}(T^0)$.

We now turn to the calculation of $\mathcal{W}_b$ in the large-T limit, namely of the diagrams with a single vector exchange and a self-energy correction $\mathcal{O}(g^2)$ ("bubble" diagrams). We indicate with B_{ij} the diagram in which the propagator, given by eqs. (2.7), (2.8) and (2.9), connects the sides γ_i, γ_j. Among all B_{ij}'s, in the limit $T\to\infty$, leading contributions are produced by $B_{11}=B_{33}$ and B_{13}; however they turn out to be only $\mathcal{O}(T)$. Those diagrams are represented in figure 6. Nonetheless there are subtleties due to the fact that the planar part of the self-energy is UV divergent;

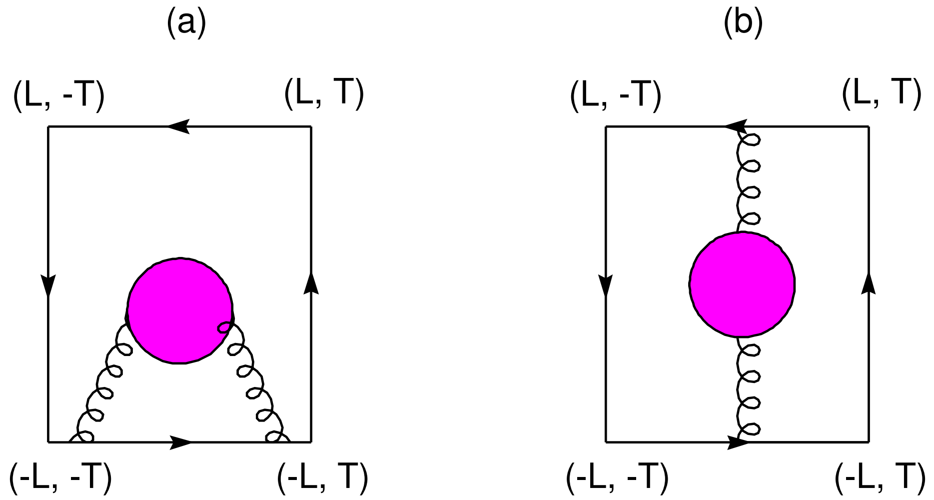


Figure 6: Dominant diagrams with a one-loop self-energy insertion: B_{33} in (a) and B_{13} in (b).

henceforth, when inserted in the Wilson loop, the latter needs being regularized. The same problem clearly occurs also in the ordinary case and is usually dealt with by considering the analytic continuation of the self-energy in the complex ω-plane and by performing the limit $T \to \infty$ while keeping $\omega \neq 2$ [27]. Once this point has been made clear, B_{33} reads

$$B_{33} = \frac{(g\,\mu^{2-\omega})^2}{(2\pi)^{2\omega}} \int \frac{d^{2\omega}p}{p^4} \int_{-T}^{T} dx \int_{x}^{T} dy\, e^{ip_0(x-y)} \left(p^2 - p_0^2\right) \Pi_1\,. \tag{4.10}$$

After integrating out x and y, one can show that, in the large-T limit, B_{33} does not increase faster than T.

In the same philosophy of ref. [27], such a contribution, divergent as $\omega \to 2$ but subleading in T, is discarded. The same large-T behaviour is exhibited by B_{13}, the only essential difference being the phase factor $\exp(2ip_3L)$, independent of the geometrical variables, in the integrand.

With the choice of non-commutative parameter we have hitherto considered ($\theta_{12} \neq 0$), the peculiar extra structure encoded in Π_2 has not been probed. A more intriguing situation occurs if we scrutinize the case $\theta_{23} \neq 0$. In this case the UV-IR mixing phenomenon, typical of a non-commutative theory, affects the $\mathcal{O}(g^4)$ loop calculation. As a matter of fact, Π_2 diverges as $(p_\perp^2)^{-2}$ in the IR, which is the counterpart of the would-be quadratic mass singularity coming from a tadpole. If we again adopt the philosophy of keeping $\omega \neq 2$, the singularity is not exposed and we recover a sub-leading behaviour in the large T-limit. However, there is an additional problem with respect to the previous case, in as much as we have to invoke continuation in ω in order to regularize both the UV and the IR behaviours.

One could assume a different attitude, namely consider a one-loop resummed vector propagator, according to eq. (3.3), keeping the dependence on the renormalization procedure, which should hopefully be irrelevant to the final result. Then, of course, IR singularities get sterilized; nonetheless, this choice should be consistently performed in all the diagrams we have considered and would imply an infinite partial resummation of perturbative diagrams, leading thereby to a result beyond $\mathcal{O}(g^4)$. In addition, one would face the troublesome problem of the appearance of the tachyonic pole we have described in the previous section, whose extra contribution could hardly be interpreted.

As a final remark, we stress that our treatment of the Wilson loop applies to a non-commutative theory of a "magnetic" type. In the "electric" case, leaving aside the difficulties related to the interpretation of such a Wilson loop in this situation, its large-T behaviour involves the non-commutative parameter via the Moyal phase. Not only one encounters considerable technical difficulties in performing such a computation, but also different scaling limits lead to different outcomes and all of them still call for a sensible explanation.

5. Conclusions

We discussed the quantum consistency of non-commutative gauge theories by investigating their unitarity properties at perturbative level. In the first part of the paper we extended the work of [6, 11] to gauge theories, studying the one-loop level analytical structure of the vacuum polarization tensor both for pure spatial (magnetic) and space-time (electric) non-commutative parameter. The general feature is a violation of Lorentz covariance through the appearance in the amplitudes of a new kinematical variable $\tilde{p}^2 = \theta^{\nu}_{\mu}\theta^{\mu}_{\lambda}p_{\nu}p^{\lambda}$; dispersion relations are strongly affected by its presence. In the magnetic case we found that Cutkoski's rules are satisfied by considering only physical branch cuts: the positivity of spectral densities related to transverse polarizations was checked, although being realized through an oscillating behaviour, and the possibility of recovering the full Feynman amplitude through its imaginary part, in spite of the non-locality of the theory, was carefully discussed. On the other hand, in the electric case we saw the appearance of extra singularities on the p_0^2-plane: a threshold starting at $p^2 = -p_{\perp}^2$, with a non-positive definite discontinuity, suggests the presence of tachyonic excitations carrying negative probabilities. Perturbative unitarity is therefore lost.

Next, by resumming the one-loop result, new poles for physical polarizations come into play: it is well known [4, 5] that, thanks to the IR/UV mixing, it is possible to obtain isolate poles, for energy well below the (usual) Landau singularity, i.e. for small momenta. In the pure spatial case a tachyon is found with positive probability, signalling, anyway, a perturbative instability. For a space-time non-commutative parameter the situation is much more exotic: two tachyonic poles appear for $-p_\perp^2 < p^2 < 0$ at perturbative (i.e. $\mathcal{O}(g^2)$) momenta, with different positivity properties. The ghostly one decouples as $g^2 \to 0$ while the other is turned, in the same limit, into a correction of the free pole. Above a certain value of the coupling both poles migrate into the complex plane.

In the second part of the paper, we proposed an extension of the usual test of time exponentiation of a Wilson loop to the non-commutative gauge theories: the definition of the loop through non-commutative path-ordering makes its physical interpretation not straightforward even in the pure spatial case. We nevertheless showed in the magnetic case that exponentiation persists at $\mathcal{O}(g^4)$, in spite of the presence of Moyal phases and of the appearance of new infrared singularities. The approximations we employed in the computation are likely to be justified in the spatial case while for space-time non-commutativity they are likely to be invalid, posing a difficulty of principle to the investigation.

Our results fit well with the common wisdom derived from the stringy picture, relating the magnetic case to a good field theoretical limit; the non-unitarity of the electric case has to be ascribed, instead, to the impossibility of decoupling massive open-string states from the light degrees of freedom. Many questions remain, nevertheless, to be answered. The first one concerns the presence of a tachyonic pole in the "magnetic" theory, that seems to signal an instability of the perturbative vacuum. It would be very interesting to study its effect on the four-point function, where unitarity poses strong constraints through the presence of crossed channels. The very same singularity seems to be an obstacle for the renormalizability program: in particular the recent proposal [21] to define an IR safe perturbation theory through resummation appears in conflict with the presence of a tachyonic pole. A more careful investigation of the vacuum properties of non-commutative gauge theories is probably needed. On the other hand it would be important to better understand if the Wilson loop test is fully justified in the non-commutative context. Wilson lines have a natural interpretation when coming from string theory [15], and therefore their relation with unitarity is likely to be simpler in that context.

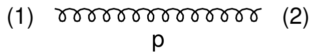
All these issues are currently under investigations.

A. Feynman rules for the non-commutative U(N) Yang-Mills theory

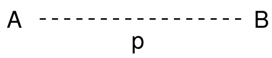
The non-commutative Yang-Mills action in the 't Hooft-Feynman gauge including ghosts takes the following form

$$S = \int d^4x \,\mathrm{Tr}\left(-\frac{1}{2g^2}F^{\mu\nu} \star F_{\mu\nu} + (\partial^\mu A_\mu)^2 - \bar{c} \star \partial^\mu D_\mu c + \partial^\mu D_\mu c \star \bar{c}\right). \qquad \text{(A.1)}$$

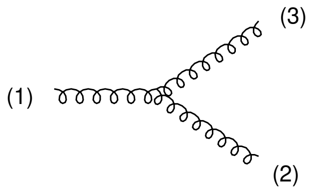
In the Feynman-'t Hooft gauge, the Feynman rules are



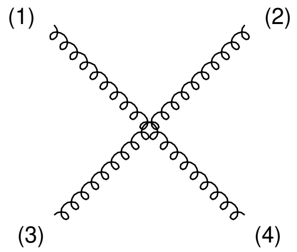
$$-\frac{i}{p^2}\,\delta^{AB} g^{\mu\nu}\,,$$



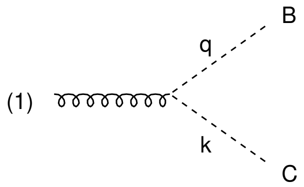
$$\frac{i}{p^2}\,\delta^{AB}\,,$$



$$2g\left(-i\cos\left(\frac{\tilde{p}q}{2}\right)\mathrm{Tr}[t^A,t^B]t^C+\sin\left(\frac{\tilde{p}q}{2}\right)\mathrm{Tr}\{t^A,t^B\}t^C\right)\times$$
$$\times\left[(k-p)^\nu g^{\mu\rho}+(p-q)^\rho g^{\mu\nu}+(q-k)^\mu g^{\nu\rho}\right],$$



$$-2ig^2\,\mathrm{Tr}\left[\left(-i\cos\left(\frac{\tilde{p}q}{2}\right)\,[t^A,t^B]+\sin\left(\frac{\tilde{p}q}{2}\right)\,\{t^A,t^B\}\right)\times\right.$$
$$\left.\times\left(-i\cos\left(\frac{\tilde{k}l}{2}\right)\,[t^C,t^D]+\sin\left(\frac{\tilde{k}l}{2}\right)\,\{t^C,t^D\}\right)\right]\times$$
$$\times\left(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}\right)+(1324)+(1423)\,,$$



$$-2g\,q^\mu\left(-i\cos\left(\frac{\tilde{p}q}{2}\right)\mathrm{Tr}[t^A,t^B]t^C+\right.$$
$$\left.+\sin\left(\frac{\tilde{p}q}{2}\right)\mathrm{Tr}\{t^A,t^B\}t^C\right),$$

where wavy and dotted lines denote gluons and ghosts, respectively, capital letters $\mathrm{U}(N)$ indices, small letters momenta. Finally we set $(1)\equiv(A,\,p,\,\mu)$, $(2)\equiv(B,\,q,\,\nu)$, $(3)\equiv(C,\,k,\,\rho)$, $(4)\equiv(D,\,l,\,\sigma)$.

We use hermitean gauge-group generators t^A with the normalization $\mathrm{Tr}(t^A\,t^B)=\frac{1}{2}\delta^{AB}$.

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